Math 251 Midterm 1 Sample

Name: ____________________

This exam has 10 questions, for a total of 100 points.

Please answer each question in the space provided. You need to write **full solutions**. Answers without justification will not be graded. Cross out anything the grader should ignore and circle or box the final answer.

Question	Points	Score
1	10	
2	15	
3	10	
4	10	
5	10	
6	10	
7	5	
8	10	
9	10	
10	10	
Total:	100	

Question 1. (10 pts)

In the following, $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ are nonzero vectors in $\mathbb{R}^3$.

(a) Does the expression $\mathbf{a} \times (\mathbf{b} \bullet \mathbf{c})$ make sense?

> **Solution:** No, it does not make sense.

(b) If $\mathbf{a} \times \mathbf{b} = \mathbf{0}$, what is the angle between $\mathbf{a}$ and $\mathbf{b}$? List all possibilities.

> **Solution:** $\mathbf{a} \times \mathbf{b} = \mathbf{0}$ implies that $\mathbf{a}$ and $\mathbf{b}$ are parallel. So the angle between $\mathbf{a}$ and $\mathbf{b}$ is either 0 or π.

(c) Given the surface $z^2 + x^2 - y^2 - 6z + 2x + 6 = 0$, determine its type:

A. ellipsoid

B. elliptic paraboloid

C. cone

D. hyperboloid of one sheet

E. hyperboloid of two sheets

Question 2. (15 pts)

(a) Find an equation of the plane that passes through the point $(3, 3, 1)$ and is orthogonal to the line

$$x = t, \quad y = 2 + t, \quad z = 3t.$$

> **Solution:** The plane has a normal vector
>
> $$\langle 1, 1, 3 \rangle$$
>
> So an equation of the plane is
>
> $$(x - 3) + (y - 3) + 3(z - 1) = 0$$
>
> equivalently,
>
> $$x + y + 3z = 9$$

(b) Find the angle between the plane in part (a) with the plane $x - y - 3z = 1$

> **Solution:** The angle between two planes is the (acute) angle between their normal vectors:
>
> $$\mathbf{u} = \langle 1, 1, 3 \rangle$$
>
> $$\mathbf{v} = \langle 1, -1, -3 \rangle$$
>
> Use the formula
>
> $$\mathbf{u} \bullet \mathbf{v} = \|\mathbf{u}\| \|\mathbf{v}\| \cos \theta$$
>
> So
>
> $$\theta = \cos^{-1}\left(\frac{9}{11}\right)$$
>
> **Remark: Note that the answer is** $\cos^{-1}\left(\frac{9}{11}\right)$, **not** $\cos^{-1}\left(\frac{-9}{11}\right)$

(c) Find the line of intersection of the plane in part (a) with the plane $x - y - 3z = 1$.

Solution: To find the line of intersection, we need to find a point on the line and the direction of the line. The direction of the line is given by

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 3 \\ 1 & -1 & -3 \end{vmatrix} = \langle 0, 6, -2 \rangle$$

To find a point on the line, we solve the following system:

$$\begin{cases} x + y + 3z = 9 \\ x - y - 3z = 1 \end{cases}$$

We get $x = 5, y = 1, z = 1$ is a solution. So $(5, 1, 1)$ is a point on the line of intersection.

So the line of intersection has the following equations

$$x = 5, \quad y = 1 + 6t, \quad z = 1 - 2t$$

Question 3. (10 pts)

A curve is described by the vector function $\mathbf{r}(t) = \langle \sin \pi t, \sqrt{t}, \cos \pi t \rangle$.

(a) Find the derivative of $\mathbf{r}(t)$.

Solution:

$$\mathbf{r}'(t) = \left\langle \pi \cos(\pi t), \quad \frac{1}{2\sqrt{t}}, \quad -\pi \sin(\pi t) \right\rangle.$$

(b) Find the tangent line to this curve at the point $(0, 1, -1)$.

Solution: First we need to figure out for which value of t the point $(0, 1, -1)$ occurs, that is, to solve

$$\langle \sin \pi t, \sqrt{t}, \cos \pi t \rangle = \langle 0, 1, -1 \rangle$$

We get $t = 1$.

Now plug $t = 1$ into the derivative from part (a), we get the tangent vector of the curve at the point $(0, 1, -1)$.

$$\mathbf{v} = \langle -\pi, 1/2, 0 \rangle$$

So the tangent line has equations:

$$x = -\pi t, \quad y = 1 + \frac{1}{2}t, \quad z = -1$$

Question 4. (10 pts)

Determine whether the following limit exists or not. Show work!

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$$

Solution: Case 1: along the x-axis, that is, $y=0$. We have

$$\lim_{x\to 0} 0 = 0$$

Case 2: along the direction $y=x$. We have

$$\lim_{x\to 0} \frac{x^2}{x^2+x^2} = \lim_{x\to 0} \frac{1}{2} = \frac{1}{2}$$

Observe that $0 \neq \frac{1}{2}$. So we conclude that the limit does not exist.

Question 5. (10 pts)

Let $z = 5x^2y + y$ with $x = s\cos t$ and $y = s^2 + e^t$. Find the value of $\dfrac{\partial z}{\partial t}$ for $(s,t) = (1,0)$.

Solution:

$$\begin{aligned}\frac{\partial z}{\partial t} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} \\ &= (10xy)(-s\cdot\sin t) + (5x^2+1)(e^t)\end{aligned}$$

For $(s,t) = (1,0)$, we get $x = 1\cdot\cos 0 = 1$ and $y = 1^2 + e^0 = 2$. Plug all these numbers into the expression of $\frac{\partial z}{\partial t}$, we have

$$\frac{\partial z}{\partial t}(1,0) = 6$$

Question 6. (10 pts)

A surface is given by an equation

$$x^2 + y^2 - 2z^2 + xyz = 2$$

Find the tangent plane of this surface at the point $(0, 2, 1)$

Solution: Set

$$F(x, yz) = x^2 + y^2 - 2z^2 + xyz - 2$$

A normal vector of the surface is given by

$$\nabla F = \langle F_x, F_y, F_z \rangle = \langle 2x + yz, 2y + xz, -4z + xy \rangle$$

So at the point $(0, 2, 1)$, we have a normal vector

$$\langle 2, 4, -4 \rangle$$

An equation of the tangent plane is given by

$$2x + 4(y - 2) - 4(z - 1) = 0.$$

Question 7. (5 pts)

Find all second partial derivatives of the function $f(x, y) = e^{x^2-y^2}$.

Solution: We need to find the first partial derivatives first.

$$\frac{\partial f}{\partial x} = e^{x^2-y^2}(2x)$$

$$\frac{\partial f}{\partial y} = e^{x^2-y^2}(-2y)$$

Now the second partial derivatives are

$$f_{xx} = e^{x^2-y^2}(2x)(2x) + e^{x^2-y^2}(2) = e^{x^2-y^2}(4x^2+2)$$

$$f_{xy} = f_{yx} = e^{x^2-y^2}(-4xy)$$

$$f_{yy} = e^{x^2-y^2}(-2y)(-2y) + e^{x^2-y^2}(-2) = e^{x^2-y^2}(4y^2-2)$$

Question 8. (10 pts)

Given the equation $xe^z = y^2 \sin(xyz) + 1000$, find $\partial z/\partial y$ by using implicit differentiation.

Solution: Set

$$F(x, y, z) = xe^z - y^2 \sin(xyz) - 1000$$

Then we have

$$\frac{\partial z}{\partial y} = \frac{-(\partial F/\partial y)}{(\partial F/\partial z)} = \frac{2y \sin(xyz) + y^2 \cos(xyz)xz}{xe^z - y^2 \cos(xyz)xy}$$

Question 9. (10 pts)

Given the function

$$z = \sqrt{y^2 - x}$$

(a) Find the gradient of the function

Solution: The gradient is

$$\nabla z = \langle \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y} \rangle = \left\langle \frac{-1}{2\sqrt{y^2 - x}}, \frac{y}{\sqrt{y^2 - x}} \right\rangle$$

(b) Find the maximum rate of change of the function at the point $(5, 3)$, and determine in which direction this maximum occurs.

Solution: The maximum rate of change at $(5, 3)$ is

$$\|\nabla z(5, 3)\| = \|\langle -1/4, 3/2 \rangle\| = \frac{\sqrt{37}}{4}$$

This occurs in the direction

$$\mathbf{u} = \frac{1}{\sqrt{37}/4}\langle -1/4, 3/2 \rangle = \langle \frac{-1}{\sqrt{37}}, \frac{6}{\sqrt{37}} \rangle$$

Question 10. (10 pts)

Use differentials to approximate the number $\sqrt{3.96}\ln(1.07)$.

Solution: Set the function

$$f(x,y) = \sqrt{x}\ln y$$

We shall compare $f(3.96, 1.07) = \sqrt{3.96}\ln(1.07)$ with

$$f(4,1) = \sqrt{4}\ln(1) = 0$$

Compute the differential of $f(x,y)$

$$df = f_x dx + f_y dy = (\frac{1}{2\sqrt{x}}\ln y)dx + \frac{\sqrt{x}}{y}dy$$

At the point $(4,1)$, we have

$$f_x(4,1) = 0, f_y(4,1) = 2$$

Moreover, $dx = 3.96 - 4 = -0.04$ and $dy = 1.07 - 1 = 0.07$. So we have

$$df = 2(0.07) = 0.14$$

Therefore,

$$\sqrt{3.96}\ln(1.07) \approx f(4,1) + df = 0 + 0.14 = 0.14$$